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Duality of two-point functions for confining potentials

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An analog to the scattering matrix describes the spectrum and high-energy behavior of confined systems. We show that for nonrelativistic systems this $\hat{S}$ matrix is identical to a two-point function which transparently describes the bound states for all angular momenta. Confined systems can thus be described in a dual fashion. This result makes it possible to study the modification of linear trajectories (originating in a long-range confining potential) due to short-range forces which are unknown except for the way in which they modify the asymptotic behavior of the two-point function. A type of effective-range expansion is one way to calculate the energy shifts.

I. INTRODUCTION

Calculation of the spectrum of quantum chromodynamics (QCD) remains one of the central goals of particle physics. Indeed, the general problem of the calculation of the bound states of any strong-coupling quantum field theory is unsolved save for the case of static models.[1] This problem would be solved if the full analytic behavior of scattering amplitudes, or four-point functions, were known, because the poles of these amplitudes in those channels with the appropriate quantum numbers describe the bound states. This is in fact the procedure followed in using the Veneziano amplitudes in dual models[2] for a zero-width approximation. Unfortunately, the only manageable method for getting bound states in quantum field theory, the Bethe-Salpeter equation with a kernel arising from a finite set of exchanges, is inadequate in QCD, where an infinite number of bound states is expected. Calculations[3] based on truncation of the Dyson-Schwinger equations for QCD suggest confinement without providing a mechanism for calculating the spectrum.

An alternative scheme[4] avoids the scattering amplitude. The two-point function of generalized currents carrying the desired quantum numbers will have an absorptive part with peaks at the location of the corresponding peaks of the physical spectrum. The fact that two-point functions depend on only a single kinematic variable presents both an advantage and a difficulty. On the one hand their structure is much simpler and easier to calculate. On the other hand, the kind of connection between bound states of different angular momentum that manifests itself in scattering amplitudes via the analytic behavior in angular momentum is, on the surface at least, absent from the two-point function. Thus it is, for example, not clear how the structure of a spin-1 two-point function

$$i\int d^4x\, e^{ik\cdot x}\langle 0\,|\,T(j_\mu(x)j_\nu(0))\,|\,0\rangle,\quad j_\mu=\bar{u}\gamma_\mu u\ ,$$

and the spin-2 two-point function

$$i\int d^4x\, e^{ik\cdot x}\langle 0\,|\,T(j_{\mu\nu}(x)j_{\alpha\beta}(0))\,|\,0\rangle\ ,$$

$$j_{\mu\nu}=\bar{u}(\gamma_\mu\overleftrightarrow{\partial}_\nu+\gamma_\nu\overleftrightarrow{\partial}_\mu-\tfrac{1}{2}g_{\mu\nu}\overleftrightarrow{\nabla\!\!\!/})u\ ,$$

are related. Nevertheless, such a connection must exist; in this paper we shall show how in potential theory for a confining potential there is a single function which describes *all* the bound-state poles.

In studying duality, we are interested in connecting the *asymptotic* behavior of the two-point function to the positions of bound-state poles. This connection is made through consideration of two functions. The first is the nonrelativistic quantum-mechanical analog of the two-point function itself, which is a spectral sum over the eigenstates. A special case of this function has been considered by Vainshtein, Zakharov, Novikov, and Shifman[5] in the context of the nonrelativistic version of the QCD sum rules,[6] and by Bell and Bertlmann.[7] The most thorough examination of the validity of these sum rules has been carried out by Durand, Durand, and Whitenton.[8] The two-point function provides direct information on the bound states. The second function is the analog of the scattering matrix for a confining potential, namely the coefficient of the regular solution to Schrödinger's equation in a solution which vanishes at large distances. This quantity is simple to calculate at large momentum. One of the central results of this paper is the establishment of the identity of the two-point function and the S matrix for confined systems, allowing for an easier study of the connection between the asymptotic behavior of the S matrix and the location and residues of the bound states of the problem.

The plan of this paper is as follows. Section II is divided into three parts which contain, respectively, definition of the "S matrix" $\hat{S}$ for a confining potential, discussion of the two-point function Π, treatment of an example, the harmonic oscillator, and a more general derivation of the identity of the two functions. In Sec. III we show how understanding of the asymptotic series for $\hat{S}$ can be used to extract any short-range terms in the potential and hence the (low-lying) bound states upon which this piece of the potential has an influence. Since only the leading few terms of the asymptotic behavior are ever known in practice, we develop an ansatz for dealing with them, which takes into account crude features of the short-range potential, as seen through the leading asymptotic behavior. This is a kind of effective-range theory, and we discuss it in Sec. IV. Section V contains some conclusions.

In previous work we have shown[4] how one can develop linear trajectories starting with a naive bag model as a zeroth-order approximation. In this work our attitude is to take linear confinement, i.e., confinement which gives linear Regge trajectories, as a starting point and to calculate the effects of short-range forces. In fact, as this work indicates and as we show in forthcoming work[9] with a detailed example, the confining part of the potential *must* be independently known if the spectrum is to be determined from limited asymptotic information. Although QCD is surely not describable by a local potential, the present study has direct relevance to QCD because in that theory too there is a confining interaction and a perturbative short-range modification, because the duality principle discussed here is analogous to those used in QCD sum rules, and also because effective potentials have been so useful in quarkonium phenomenology.

II. THE SCATTERING MATRIX AND THE TWO-POINT FUNCTION

A. "Scattering matrix" for a confining potential

The radial Schrödinger equation

$$\frac{d^2u}{dr^2}-\frac{\nu^2-\frac{1}{4}}{r^2}u+k^2u=V(r)u\ , \tag{2.1}$$

where $\nu=l+\frac{1}{2}$, has unrenormalized solutions $u(\nu,r,k)$ (in the following we often suppress the argument k) behaving near $r=0$ like[10]

$$u(\nu,r)=r^{\nu+1/2}[1+O(r^2)]\ . \tag{2.2}$$

For ν positive this solution is regular, and for ν negative it is irregular. We conventionally take ν positive in the following, so that $u(\pm\nu,r)$ are regular/irregular solutions. For $V(r)$ confining, $V(r)\to_{r\to\infty}\infty$, the spectrum is purely discrete, $E_n=k_n{}^2$. The solution which vanishes at a large $r=R$ is

$$\bar{u}(\nu,r;R)=u(-\nu,r)-\frac{u(-\nu,R)}{u(\nu,R)}u(\nu,r)\ , \tag{2.3}$$

and it is an eigensolution if it is regular at the origin. This is possible when the coefficient of $u(\nu,r)$ has a pole. We see that the function

$$\hat{S}(\nu,E)=\lim_{R\to\infty}\frac{u(-\nu,R)}{u(\nu,R)} \tag{2.4}$$

can be regarded as the generalization of the S matrix for a confining potential. It obeys a form of unitarity, namely,

$$\hat{S}(\nu,E)\hat{S}(-\nu,E)=1\ . \tag{2.5}$$

As an example we can consider the harmonic oscillator, with $V(r)=\frac{1}{4}\omega^2r^2$, whose regular solution is

$$u(\nu,r)=r^{\nu+1/2}e^{-\omega r^2/4}\Phi\left[-\frac{\eta}{2}+\frac{\nu+1}{2},\nu+1,\tfrac{1}{2}\omega r^2\right], \tag{2.6}$$

where Φ is the confluent hypergeometric function, $\eta\equiv E/\omega$, and we have chosen mass $=\frac{1}{2}$. For large R ($R^2\gg\omega^{-1}$),

$$u(\nu,R)\approx\frac{\Gamma(\nu+1)}{\Gamma(-\eta/2+(\nu+1)/2)}\left[\frac{\omega}{2}\right]^{-(\eta+\nu+1)/2}\times R^{-\eta-1/2}e^{\omega R^2/4} \tag{2.7}$$

and hence

$$\hat{S}=\left[\frac{\omega}{2}\right]^{+\nu}\frac{\Gamma(-(\eta-1)/2+\nu/2)\Gamma(1-\nu)}{\Gamma(-(\eta-1)/2-\nu/2)\Gamma(1+\nu)}\ . \tag{2.8}$$

This function has poles when

$$-\frac{\eta}{2}+\frac{1}{2}+\frac{\nu}{2}=-m,\quad m=0,1,2,\ldots$$

or

$$E=\omega(2m+\nu+1)=\omega(2m+l+\tfrac{3}{2})\ . \tag{2.9}$$

[The additional fixed poles in ν at (unphysical) integer values of ν are associated with the positivity of the residues of the moving poles in E.] Another example is the spherical well of radius R. There (see Sec. IV)

$$\hat{S}(\nu)=\left[\frac{k}{2}\right]^{2\nu}\frac{J_{-\nu}(kR)}{J_\nu(kR)}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \tag{2.10}$$

with poles at the zeros of $J_\nu(kR)$.

An alternative form for $\hat{S}$ follows from consideration of the Wronskian $W(u(\nu,r),u(-\nu,r))$, which for two solutions of the Schrödinger equation (2.1) must have the constant value -2ν:

$$-2\nu=u(\nu,r)u'(-\nu,r)-u'(\nu,r)u(-\nu,r)\ . \tag{2.11}$$

Thus

$$\frac{d}{dr}\frac{u(-\nu,r)}{u(\nu,r)}=-\frac{2\nu}{u^2(\nu,r)} \tag{2.12}$$

and by integration

$$\frac{u(-\nu,R)}{u(\nu,R)}=\frac{u(-\nu,\epsilon)}{u(\nu,\epsilon)}-2\nu\int_\epsilon^R\frac{dr}{u^2(\nu,r)}\ ,$$

or, using Eq. (2.2),

$$\hat{S}(\nu)=-2\nu\int_0^\infty dr\left[\frac{1}{u^2(\nu,r)}-\frac{1}{r^{2\nu+1}}\right]. \tag{2.13}$$

The second term in the integral serves only to regularize the integral and cannot give rise to poles in $\hat{S}(\nu)$. Equation (2.13) shows directly that if $u(\nu,r)$ vanishes at infinity, $\hat{S}$ will indeed have poles.

Equation (2.13) is of course consistent with the general condition for high-lying bound states in the WKB approximation. Let us write

$$\bar{V}(r)=\frac{\nu^2-\frac{1}{4}}{r^2}+V(r)\ ;$$

$\bar{V}(r)$ is the effective potential. Between the turning points A and B, $A<B$, defined by

$$E=\bar{V}(A)=\bar{V}(B)\ ,$$

the regular wave function is

$$u(\nu,r)=[E-\bar{V}(r)]^{-1/4}\cos g(r)\ ,$$
$$g(r)=\int_A^r dr'[E-\bar{V}(r')]^{-1/2}-\frac{\pi}{4}\ . \tag{2.14}$$

We use this form in the relevant part of Eq. (2.13) to find

$$\hat{S}=-2\nu\int_A^B dr\frac{dg}{dr}\frac{1}{\cos^2 g(r)}$$
$$=-2\nu[\tan g(B)-\tan g(A)]\ . \tag{2.15}$$

This function has poles when $g(B)=(n+\frac{1}{2})\pi$, or

$$\int_A^B dr'[E-\bar{V}(r')]^{1/2}=(n+\tfrac{3}{4})\pi\ . \tag{2.16}$$

For large n this is the usual WKB eigenvalue condition.

B. Nonrelativistic two-point function

In this part we consider eigenfunctions of the Schrödinger equation (2.1), denoted by $u_n(\nu,r)$. These behave at $r=0$ like

$$u_n(\nu,r)=a_n(\nu)r^{\nu+1/2}[1+O(r^2)]\ , \tag{2.17}$$

and are orthonormal,

$$\int_0^\infty dr\, u_n(\nu,r)u_m(\nu,r)=\delta_{mn}\ .$$

The index n specifies the energy eigenvalue. These functions differ from the $u(\nu,r)$ in that they are fully normalized,

$$\int_0^\infty dr\,|u_n(\nu,r)|^2=1\ ,$$

so that

$$u_n(\nu,r)=a_n(\nu)u(\nu,r,k=k_n)\ . \tag{2.18}$$

We define the two-point function for fixed ν by

$$\Pi(E,\nu)=\sum_{n=0}^\infty\frac{2\nu[a_n(\nu)]^2}{E_n-E}\ . \tag{2.19}$$

The factor $2\nu=2l+1$ is a statistical weight for the "width factor" that appears in the numerator. For $l=0$ the spectral sum is just[11]

$$\Pi_0(E)=\sum_{n=0}^\infty\frac{|R_n(0)|^2}{E_n-E}\ , \tag{2.20}$$

where

$$R_n(0)=(1/r)u_n(\tfrac{1}{2},r)\,|_{r\to 0}$$

is the radial part of the wave function at the origin. Moments of Eq. (2.20) have been of recent interest: The nonrelativistic version of the QCD sum rules of Vainshtein *et al.*[5] involves

$$M(p,E)=\sum_{n=0}^\infty\frac{|R_n(0)|^2}{(1+(E_n/E)^{p+1}}$$
$$=E^p\frac{1}{p!}\left[\frac{d}{dE}\right]^p\Pi_0(E)\Big|_{E<0}\ . \tag{2.21}$$

These have been studied in the Born approximation by the authors of Ref. 5 and more extensively by Bell and Bertlmann.[7]

C. Connection between $\hat{S}$ and Π

We establish the identity of these two functions in this part, beginning with the example of the harmonic oscillator. By normalizing Eq. (2.6) and using Eq. (2.9) for the eigenvalues we find

$$[a_n(\nu)]^2=2\left[\frac{\omega}{2}\right]^{\nu+1}\frac{\Gamma(\nu+n+1)}{\Gamma(n+1)\Gamma^2(\nu+1)}\ . \tag{2.22}$$

Thus

$$\Pi(E,\nu)=2\left[\frac{\omega}{2}\right]^{\nu}\sum_{n=0}^\infty\frac{\Gamma(\nu+n+1)}{\Gamma(n+1)\nu\Gamma^2(\nu)}\frac{1}{2n+\nu+1-\eta}\ . \tag{2.23}$$

To obtain the required sum, we rewrite the last equation as

$$\Pi(E,\nu)=\left[\frac{\omega}{2}\right]^{\nu}\frac{2}{\Gamma(\nu)(\nu+1-\eta)}\sum_{n=0}^\infty\frac{\Gamma((\nu+1-\eta)/2+1)\Gamma(\nu+1+\eta)\Gamma((\nu+1-\eta)/2+n)}{\Gamma((\nu+1-\eta)/2+1+n)\Gamma(\nu+1)\Gamma((\nu+1-\eta)/2)}\frac{1}{n!}\ .$$

The sum is now recognized as a hypergeometric series:

$$\Pi(E,\nu)=\left[\frac{\omega}{2}\right]^{\nu}\frac{2}{\Gamma(\nu)(\nu+1-\eta)}\,{}_2F_1\left[\nu+1,\frac{\nu+1-\eta}{2};\frac{\nu+1-\eta}{2}+1;1\right]\ . \tag{2.24}$$

Thus

$$\Pi(E,\nu)=\left[\frac{\omega}{2}\right]^{\nu}\frac{\Gamma(-\nu)\Gamma(1-\eta+\nu)/2)}{\Gamma(\nu)\Gamma((1-\eta-\nu)/2)} . \tag{2.25}$$

Up to an (uninteresting) factor of -1, this is precisely $\hat{S}$ for the harmonic oscillator, Eq. (2.8). Note that while formally the sum (2.23) does not exist,[5,7] its development through Eq. (2.25) defines it by analytic continuation.

To demonstrate this result more generally we relate two forms for the Green's function $G_E(r,r')$ for the Schrödinger operator

$$\mathscr{L}(r)\equiv\frac{d^2}{dr^2}-\frac{\nu^2-\frac{1}{4}}{r^2}-V(r) , \tag{2.26}$$

satisfying

$$[\mathscr{L}(r)+E]G_E(r,r')=-\delta(r-r') . \tag{2.27}$$

On the one hand,

$$G_E(r,r')=\sum_n\frac{u_n(\nu,r)u_n(\nu,r')}{E_n-E} , \tag{2.28}$$

where $u_n(\nu,r)$ are the eigenfunctions [Eq. (2.18)], and on the other hand,

$$G_E(r,r')=+\frac{1}{2\nu}[u(\nu,r)\bar{u}(\nu,r';R)\theta(r'-r)$$
$$+u(\nu,r')\bar{u}(\nu,r;R)\theta(r-r')] , \tag{2.29}$$

where the $u(\nu,r)$ are the regular solutions, as in Eq. (2.2), and the $\bar{u}(\nu,r;R)$ are the solutions which vanish at R [Eq. (2.3)]. For $R\to\infty$ and $r\le r'$,

$$\sum_n\frac{u_n(\nu,r)u_n(\nu,r')}{E_n-E}$$
$$=+\frac{1}{2\nu}u(\nu,r)[u(-\nu,r')-\hat{S}(\nu,E)u(\nu,r')] . \tag{2.30}$$

We multiply Eq. (2.30) by $r^{-\nu-1/2}$ and let $r\to 0$, then multiply the result by $r'^{-\nu-1/2}$ and let $r'\to 0$. From Eqs. (2.2) and (2.18) we find

$$\sum_n\frac{[a_n(\nu)]^2}{E_n-E}=\frac{1}{2\nu}\lim_{r'\to 0}[r'^{-2\nu}-\hat{S}(\nu,E)] . \tag{2.31}$$

For negative ν the right-hand side exists and is $-(1/2\nu)\hat{S}(\nu,E)$. To understand the left-hand side we need to understand the convergence of the series for negative ν. We first note that the particle becomes free in a box of dimension R, the parameter of $\hat{S}$, when $n\to\infty$. In other words, for large n

$$u_n(\nu,r)\to A_n(\nu)rj_{\nu-1/2}(k_nr) , \tag{2.32}$$

where k_n is determined by

$$j_{\nu-1/2}(kR)=(\pi/2kR)^{1/2}J_\nu(kR)=0 ,$$

and $E_n=k_n{}^2$. $A_n(\nu)$ is determined from Eq. (2.32) by the normalization condition

$$1=A_n{}^2(\nu)\int_0^\infty dr\, r^2[j_{\nu-1/2}{}^2(k_nr)]^2$$
$$=A_n{}^2(\nu)\frac{\pi R^2}{2k_n}\int_0^1 d\rho\,\rho J_\nu{}^2(k_nR\rho) . \tag{2.33}$$

The last integration can be performed if $\nu\ge -1$ and gives

$$A_n{}^2(\nu)=\frac{4k_n}{\pi R^2}\frac{1}{[J_\nu'(k_nR)]^2} ,\quad \nu\ge -1 . \tag{2.34}$$

We assume $A_n{}^2(\nu)$ is defined for negative ν by analytic continuation from Eq. (2.34). Note that since $J_\nu(k_nR)=0$,

$$J_\nu'(k_nR)=J_{\nu-1}(k_nR)=J_{\nu+1}(k_nR) .$$

For large values of n and fixed ν, the nth zero of J_ν, k_nR, behaves like

$$k_nR\sim(n+\tfrac{1}{2}\nu-\tfrac{1}{4})\pi+O\left[\frac{1}{n}\right]$$
$$\neq[n'+\tfrac{1}{2}(\nu\pm 1)-\tfrac{1}{4}]\pi , \tag{2.35}$$

so that a zero of J_ν will not be a zero of $J_{\nu\pm 1}$, and $A_n{}^2(\nu)$ will have no poles for integer values of n. To find $a_n(\nu)$, we need the threshold behavior of Eq. (2.32). We get

$$a_n{}^2(\nu)=A_n{}^2(\nu)\frac{\pi}{4}\frac{1}{\Gamma^2(\nu+1)}\left[\frac{k_n}{2}\right]^{2\nu-1}$$
$$=\frac{2}{R^2}\frac{1}{[J_\nu'(k_nR)]^2}\frac{1}{\Gamma^2(\nu+1)}\left[\frac{k_n}{2}\right]^{2\nu} . \tag{2.36}$$

The asymptotic behavior of $J_\nu'(k_nR)$ for large argument is

$$\left[\frac{2}{\pi k_nR}\right]^{1/2}O(1) ,$$

where $O(1)$ means a cosine function. All in all, the summand of the left-hand side of Eq. (2.31) behaves at large n like $k_n{}^{2\nu-1}$, which gives a convergent series for negative ν. Assuming there are no natural barriers at $\nu=0$, and there are none for potential scattering, the negative-ν result can then be used to define the two-point function for positive ν. We have thus established that

$$\Pi(E,\nu)=-\hat{S}(\nu,E) \tag{2.37}$$

for confining potentials, making clear the crucial role $\hat{S}(\nu,E)$ plays in determining the bound states of a confining potential.

III. EFFECTS OF SHORT-DISTANCE PERTURBATIONS

In this section we work with the "S matrix" $\hat{S}$ defined for confining potentials in Sec. II in order to see how a short-range component to the potential affects the asymptotic behavior of $\hat{S}$ as well as the properties of the low-lying bound states. Such a component will necessarily have parameters associated with it describing the short-range scale and the strength. As we shall see, these parameters describe certain analyticity properties of the results.

We suppose

$$V(r)=V_L(r)+\delta V(r)\ , \tag{3.1}$$

where the range scale for the short-range perturbation $\delta V(r)$ will be taken as r_0. We work with the radial part of the Schrödinger equation, and with solutions satisfying

$$u''-\frac{\nu^2-\frac{1}{4}}{r^2}u+Eu=V(r)u\ , \tag{3.2}$$

$$w''-\frac{\nu^2-\frac{1}{4}}{r^2}w+Ew=V_L(r)w\ , \tag{3.3}$$

and Eq. (2.2) for the small-r behavior. We can establish by standard methods that u satisfies the integral equation

$$u(\nu,r)=w(\nu,r)+\frac{1}{2\nu}\int_0^r dr'\delta V(r')u(\nu,r')[w(\nu,r)w(-\nu,r')-w(-\nu,r)w(\nu,r')]\ . \tag{3.4}$$

Equation (3.4) can be rewritten

$$u(\nu,r)=A(\nu,r)w(\nu,r)+B(\nu,r)w(-\nu,r)\ , \tag{3.5}$$

where

$$A(\nu,r)=1+\frac{1}{2\nu}\int_0^r dr'\delta V\,u(\nu,r')w(-\nu,r')\ ,$$
$$B(\nu,r)=-\frac{1}{2\nu}\int_0^r dr'\delta V\,u(\nu,r')w(\nu,r')\ . \tag{3.6}$$

We have

$$\hat{S}(\nu)=\lim_{R\to\infty}\hat{S}(\nu,R)=\lim_{R\to\infty}\frac{u(-\nu,R)}{u(\nu,R)}\ . \tag{3.7}$$

Note that in the absence of a short-range perturbation,

$$S_0(\nu)=\lim_{R\to\infty}\frac{w(-\nu,R)}{w(\nu,R)}\ . \tag{3.8}$$

A useful parameter will be

$$\lambda(\nu)=\lim_{R\to\infty}\frac{B(\nu,R)}{A(\nu,R)}\ , \tag{3.9}$$

in terms of which

$$\hat{S}(\nu,R)=\frac{w(-\nu,R)+\lambda(-\nu)w(\nu,R)}{w(\nu,R)+\lambda(\nu)w(-\nu,R)}\frac{A(-\nu,R)}{A(\nu,R)}$$
$$\equiv\frac{\hat{N}}{\hat{D}}\frac{A(-\nu,R)}{A(\nu,R)}\ . \tag{3.10}$$

In the sense that δV is small and is not singular as $r\to 0$, the integrals in A and B, Eqs. (3.6), are well defined and small compared to unity. Thus $A(-\nu,R)/A(\nu,R)$ gives no poles, and we can concentrate on the first factor in $\hat{S}(\nu)$. In this form it is clear that λ, which is small, measures shifts in pole parameters, since the zeros of $w(\nu,R)$ in the large-R limit fix the poles of $\hat{S}_0(\nu)$.

To study this question we suppose the poles of the unperturbed system are at $E=E_n^0$, $w(\nu,R,E_n^0)=0$, so that for E near E_n^0,

$$w(\nu,E)\approx w'(\nu,E_n^0)(E-E_n^0)\ , \tag{3.11}$$

$$w(-\nu,E)\approx w(-\nu,E_n^0)+w'(-\nu,E_n^0)(E-E_n^0)\ . \tag{3.12}$$

Here and in the following we have suppressed the label R and the prime means differentiation with respect to E. We find that to $O[\lambda(\nu)]$,

$$\hat{D}\approx[w'(\nu,E_n^0)+\lambda(\nu)w'(-\nu,E_n^0)]\times\left[E-E_n^0+\lambda(\nu)\frac{w(-\nu,E_n^0)}{w'(\nu,E_n^0)}\right]\ . \tag{3.13}$$

This function has poles at

$$E_n^{(1)}=E_n^0-\lambda(\nu)\frac{w(-\nu,E_n^0)}{w'(\nu,E_n^0)}\ . \tag{3.14}$$

To find the residue, we evaluate $\hat{N}$ at this value of $E_n^{(1)}$, using (3.11) and (3.12). We then find for $\hat{S}$ the result, to $O(\lambda)$, or $O(\delta V)$,

$$\hat{S}=\frac{1}{E-E_n^0+\lambda(\nu)w(-\nu,E_n^0)/w'(\nu,E_n^0)}\frac{w(-\nu,E_n^0)}{w'(\nu,E_n^0)}\times\left[1-2\lambda(\nu)\frac{w'(-\nu,E_n^0)}{w'(\nu,E_n^0)}-\frac{1}{2\nu}\int_0^R dr\,\delta V[u(-\nu,r)w(\nu,r)+u(\nu,r)w(-\nu,r)]\right]\ . \tag{3.15}$$

The factor $w(-\nu,E_n^0)/w'(\nu,E_n^0)$ is just the residue for the case of no short-range correction; the term in large parentheses gives the effect of such a correction. The last factor in the large parentheses comes from the expansion of the factor $A(-\nu,R)/A(\nu,R)$. For small δV we should take $\lambda(\nu)\approx B(\nu)=O(\delta V)$. It is also useful to note that the last term in Eq. (3.15) can be written

$$\frac{1}{2\nu}\int_0^R dr\,\delta V[u(-\nu,r)w(\nu,r)+u(\nu,r)w(-\nu,r)]=A(\nu)-A(-\nu)\ . \tag{3.16}$$

Thus to $O(\delta V)$,

$$\hat{S}=\frac{1}{E-E_n^0+B(\nu)w(-\nu,E_n^0)/w'(\nu,E_n^0)}\frac{w(-\nu,E_n^0)}{w'(\nu,E_n^0)}\left[1-[A(\nu)-A(-\nu)]-2B(\nu)\frac{w'(-\nu,E_n^0)}{w'(\nu,E_n^0)}\right]. \tag{3.17}$$

Another form of $\hat{S}$ is useful for studying the high-energy behavior. Let us define

$$D(\nu)\equiv A(\nu)+B(\nu)\hat{S}_0(\nu). \tag{3.18}$$

Then

$$\hat{S}(\nu)=\hat{S}_0(\nu)\frac{D(-\nu)}{D(\nu)}, \tag{3.19}$$

and the factor $D(-\nu)/D(\nu)$ provides corrections to the high-energy behavior of $\hat{S}_0(\nu)$. We have from examination of Eq. (3.18)

$$D(\nu)=1+\frac{1}{2\nu}\int_0^R dr\,\delta V\,u(\nu,r)\left[w(-\nu,r)-\frac{w(-\nu,R)}{w(\nu,R)}w(\nu,r)\right]; \tag{3.20}$$

$u(\nu,r)$ should be replaced by $w(\nu,r)$ in Eq. (3.20) for δV small.

We have outlined in this section a connection between poles, residues, and asymptotic behavior, when a short-distance potential perturbs a longer-range confining potential. The results provided here form *in principle* a calculational scheme complete in itself.

Direct application of these formulas is analytically intractable for technical reasons. We can illustrate this best through an example:

$$V(r)=\tfrac{1}{4}\omega^2r^2+g\frac{e^{-\mu r}}{r}. \tag{3.21}$$

The long-range part V_L of the potential, a harmonic oscillator, gives linear Regge trajectories. The solution to the Schrödinger equation corresponding to this piece of $V(r)$ is given by Eq. (2.6), with large-r behavior of Eq. (2.7). The large-r behavior fully determines $\hat{S}_0$, but to find the $O(g)$ corrections we need to use $w(\nu,r)$ over the range of r where δV is important. Finite integrals over confluent hypergeometric functions are quite formidable. Even at large k we have from Eq. (2.6)

$$w(\nu,r)\underset{k\to\infty}{\sim} r^{1/2}\left[\frac{k}{2}\right]^{-\nu}\Gamma(\nu+\tfrac{1}{2})\times\sum_{n=0}^{\infty}A_n\left[\frac{\omega r}{2k}\right]^n J_{\nu+n}(kr), \tag{3.22}$$

where the A_n can be determined recursively by

$$A_0=1,\;A_1=0,\;A_2=\tfrac{1}{2}(\nu+1),\quad A_{n+1}=\frac{1}{n+1}\left[(n+\nu)A_{n-1}-\frac{k^2}{\omega}A_{n-2}\right]. \tag{3.23}$$

While the series (3.22), written as a power series in $1/k$, does converge, at least asymptotically, at large k, it does so very slowly. Moreover, the two integrals required in Eq. (3.20) cannot be evaluated in closed form even term by term in the expansion (3.22).

In the face of these difficulties, let us not lose sight of our ultimate aim. We are not interested *per se* in still another method of solving the Schrödinger equation. Instead, we want to be able to relate the asymptotic behavior of the two-point function—determined by perturbative calculations of field theory—to the positions of bound-state poles. We assume we know the long-range behavior, presumably linear confinement according to some definite calculation in QCD. In other words, we assume we know the asymptotic behavior of $\hat{S}_0$, say from this definitive QCD calculation. We have in previous work[4] shown how this behavior leads to a definite set of poles, e.g., linear trajectories. We may also know some corrections to the asymptotic behavior of $\hat{S}_0$, coming from a perturbative calculation of two-point functions; this behavior includes short-distance corrections. We want a scheme which gives the corresponding corrections to the pole positions and residues. Using the results of this section as a guide, we develop in the next section an ansatz which leads to the desired result.

IV. ANSATZ FOR SOLUTION

Since we cannot analytically perform the integrations of Sec. III, we attempt here to construct a form for $\hat{S}(\nu)$ under simpler assumptions, then to generalize this form to more complicated cases, taking into account constraints which follow from those simpler assumptions. Our procedure is to work out the $\hat{S}$ matrix for a particular short-range potential incorporating a strength and a range, but this time embedded in a spherical cavity rather than in a harmonic oscillator. We shall see certain analyticity properties appear which should be general. We then generalize the results to the *form* dictated by Sec. III, this time in a general long-range background potential. All of this can be adequately illustrated for the S-wave case, $\nu=\frac{1}{2}$, and we shall work this case out in some detail for the example

$$\delta V(r)=-g^2e^{-\mu r}, \tag{4.1}$$

with $V_L=0$ for $r<R$ and infinite for $r>R$.

The regular wave function corresponding to V_L is

$$w(\nu,r)=\Gamma(1+\nu)\left[\frac{2}{k}\right]^{\nu} r^{1/2}J_{\nu}(kr)\ , \tag{4.2}$$

and in particular

$$w(\tfrac{1}{2},r)=\frac{1}{k}\sin kr,\quad w(-\tfrac{1}{2},r)=\cos kr\ . \tag{4.3}$$

When we add δV to V_L, the Schrödinger equation is solved for $\nu=\pm\frac{1}{2}$ by

$$Z_{\pm 2ik/\mu}\left[\frac{2g}{\mu}e^{-\mu r/2}\right],$$

where Z is any Bessel function. In order to find the appropriate combinations giving regular and irregular solutions we require

$$u(\nu,r)\underset{r\to 0}{\sim}\begin{cases} r, & \nu=+\frac{1}{2},\\ 1, & \nu=-\frac{1}{2},\end{cases} \tag{4.4}$$

and

$$\frac{d}{dr}u(\nu=-\tfrac{1}{2},r)\underset{r\to 0}{\to}0\ , \tag{4.5}$$

since a solution must generally behave as Eq. (2.2), for nonsingular δV.[10] These conditions mean that $u(\nu,r)$ must be a linear combination of

$$J_{\pm 2ik/\mu}\left[\frac{2g}{\mu}e^{-\mu r/2}\right],$$

with relative coefficients determined by Eq. (4.4) for $\nu=+\frac{1}{2}$ and (4.5) for $\nu=-\frac{1}{2}$. This gives us

$$\begin{aligned} u(\tfrac{1}{2},r)&=\mathscr{C}_+\left[J_{2ik/\mu}\left[\frac{2g}{\mu}\right]J_{-2ik/\mu}\left[\frac{2g}{\mu}e^{-\mu r/2}\right]-J_{-2ik/\mu}\left[\frac{2g}{\mu}\right]J_{2ik/\mu}\left[\frac{2g}{\mu}e^{-\mu r/2}\right]\right],\\ u(-\tfrac{1}{2},r)&=-g\mathscr{C}_-\left[J'_{2ik/\mu}\left[\frac{2g}{\mu}\right]J_{-2ik/\mu}\left[\frac{2g}{\mu}e^{-\mu r/2}\right]-J'_{-2ik/\mu}\left[\frac{2g}{\mu}\right]J_{2ik/\mu}\left[\frac{2g}{\mu}e^{-\mu r/2}\right]\right]. \end{aligned} \tag{4.6}$$

The prime refers to differentiation with respect to the variable $y=(2g/\mu)e^{-\mu r/2}$, evaluated at $r=0$. The constants $\mathscr{C}_{\pm}$- are

$$\mathscr{C}_+=\frac{\Gamma(1+2ik/\mu)\Gamma(1-2ik/\mu)}{2ik}=-\mathscr{C}_-\ . \tag{4.7}$$

Note the correct limit

$$u(\pm\tfrac{1}{2},r)\underset{g\to 0}{\to}w(\pm\tfrac{1}{2},r)\ .$$

Note also that with the solutions (4.6) and (4.7), we automatically satisfy the equation for the Wronskian:

$$W(u(\tfrac{1}{2},r),u(-\tfrac{1}{2},r))=-2\nu=-1\ . \tag{4.8}$$

This requirement could indeed be used to determine the product $\mathscr{C}_+\mathscr{C}_-$, and it will play an important role in the generalization of our results.

We can form

$$\hat{S}=\lim_{R\to\infty}\frac{u(-\frac{1}{2},R)}{u(\frac{1}{2},R)}$$

to study its poles and large-k behavior. We are interested not in the exact results but in the expansion to lowest order in g in order to compare with the results of Sec. III. Let us begin with a reminder of $\hat{S}_0$ for the long-range portion of the potential:

$$\hat{S}_0=k\frac{\cos kr}{\sin kr}\underset{k\to\infty}{\sim}-ik\quad(\mathrm{Im}k>0)\ . \tag{4.9}$$

For $\hat{S}$, we are interested in the $O(g^2)$ expansion. To do this, we use the power-series expansion of the Bessel functions in Eq. (4.6). To $O(g^2)$ we find

$$\begin{aligned} u(\tfrac{1}{2},R)&=w(\tfrac{1}{2},R)\left[1-g^2\frac{1+e^{-\mu R}}{\mu^2+4k^2}\right]\\ &\quad+w(-\tfrac{1}{2},R)\frac{2}{\mu}\frac{g^2(1-e^{-\mu R})}{\mu^2+4k^2}\ ,\\ u(-\tfrac{1}{2},R)&=w(-\tfrac{1}{2},R)\left[1+g^2\frac{1-e^{-\mu R}}{\mu^2+4k^2}\right]\\ &\quad-w(\tfrac{1}{2},R)\frac{\mu g^2}{\mu^2+4k^2}\left[1+\frac{2k^2}{\mu^2}(1-e^{-\mu R})\right], \end{aligned} \tag{4.10}$$

where the wave functions $w(\pm\frac{1}{2},R)$ corresponding to the long-range potential are given by Eq. (4.3). Note that Eq. (4.10) is exactly the form derived in Sec. III, viz. (3.15), where

$$u(\nu,r)=w(\nu,r)[1+O(\delta V)]+w(-\nu,r)O(\delta V)\ .$$

Since by "short range" we mean $\mu R\gg 1$, we can safely drop factors $e^{-\mu R}$ in Eq. (4.10).

$\hat{S}$ can now be written in the canonical form

$$\begin{aligned}\hat{S}&\equiv\frac{u(-\frac{1}{2},R)}{u(\frac{1}{2},R)}\\ &=\frac{\hat{S}_0\left[1+\dfrac{g^2}{\mu^2+4k^2}\right]-\dfrac{k^2}{\mu}\dfrac{g^2}{\mu^2+4k^2}\left[2+\dfrac{\mu^2}{k^2}\right]}{1-\dfrac{g^2}{\mu^2+4k^2}+\hat{S}_0\dfrac{2}{\mu}\dfrac{g^2}{\mu^2+4k^2}}\ . \end{aligned} \tag{4.11}$$

Here

$$\hat{S}_0=\frac{w(-\frac{1}{2},R)}{w(\frac{1}{2},R)}=k\cot kR$$

is the $\hat{S}$ matrix for the unperturbed cavity; its large-k behavior is just $(-ik)$. At finite k, the poles of $\hat{S}$, i.e., of the perturbed cavity, can be read off from Eq. (4.11). These poles are the zeros of the denominator; the poles of $\hat{S}_0$ cancel top and bottom and the numerator has no other poles for physical k^2. Note the analytic structure at $k^2=-\frac{1}{4}\mu^2$, which is a correct reflection of the exponential potential and will play a role in our ansatz. In old S-matrix language these were left-hand singularities.

The asymptotic behavior of $\hat{S}$ is easily computed from Eq. (4.11). We denote this limit of $\hat{S}$ by $\hat{S}_A$ for convenience:

$$\hat{S}_A=\lim_{k\to\infty}(-ik)\left[1+\frac{2g^2}{\mu^2+4k^2}\left[1+\frac{\mu}{2ik}\right]+O(g^4)\right]$$
$$=(-ik)\left[1+\frac{g^2}{2k^2}+\frac{g^2\mu}{4ik^3}+\frac{g^2\mu^2}{8k^4}+\cdots+O(g^4)\right]. \tag{4.12}$$

The factor $(-ik)$ is the full asymptotic behavior of $\hat{S}_0$, up to exponentials in k. If an experimental measurement of a two-point function were involved, only $\mathrm{Im}\hat{S}_A$ would count. We see by examining the first few terms of (4.12) that if we knew $\hat{S}_0$, then we could deduce the parameters of the short-range perturbation from the asymptotic series, and even deduce whether the short-range perturbation has a short-distance expansion identical to the expansion of the exponential. This follows because[12] powers of a potential r^n correspond to asymptotic behavior in $\hat{S}$ in the form $a/k^{n+2}+b/k^{n+3}+\ldots$. Here the $O(r^0)$ term in an expansion of $e^{-\mu r}$ leads to a first correction of $O(k^{-2})$.

We can now make an ansatz for the general large-R form of the wave function to $O(g^2)$ in a general long-range background, namely,

$$u(\nu,R)=w(\nu,R)\left[1+F(\nu)\frac{g^2}{\mu^2+4k^2}\right]$$
$$+w(-\nu,R)(+k^2)^{-\nu}\frac{(+k^2)^{1/2}}{\mu}G(\nu)\frac{g^2}{\mu^2+4k^2}, \tag{4.13}$$

where for large k

$$F(\nu)=\sum_{n=0}f_n(\nu)\left[\frac{\mu^2}{k^2}\right]^n,\quad G(\nu)=\sum_{n=0}g_n(\nu)\left[\frac{\mu^2}{k^2}\right]^n. \tag{4.14}$$

This gives a generalization of $\hat{S}$ corresponding to Eq. (4.11). In our ansatz we have incorporated the singularity structure at $k^2=-\frac{1}{4}\mu^2$ of Eq. (4.10); this structure should reflect δV alone. The choice of the f_i and g_i depends entirely on the short-distance expansion of the potential. Given that the leading short-range terms in r come from the perturbing potential alone, the coefficients f_i and g_i to a corresponding value of i could then be determined by seeing the effect of that short-range potential in a cavity.[4] For example, we have found for the exponential potential at $\nu=\frac{1}{2}$

$$g_0(\tfrac{1}{2})=2=g_0(-\tfrac{1}{2}),\quad g_1(\tfrac{1}{2})=0,\quad g_i(-\tfrac{1}{2})=1,$$
$$f_0(\tfrac{1}{2})=-1=-f_0(-\tfrac{1}{2}),\quad f_1(\tfrac{1}{2})=0=f_1(-\tfrac{1}{2}).$$

We have included the factor $(+k^2)^\nu$ in (4.13) because we know the general asymptotic form of $\hat{S}(\nu)$ leads[4] with $(+k^2)^\nu$. This is in accordance with Eq. (4.10). Also, we know[4] that if δV is nonsingular at the origin, then the asymptotic behavior of $\hat{S}$ will have structure $\sim(+k^2)^\nu[1+O(1/k^2)]$, i.e., there is no $O(1/k)$ term. This requires us to have

$$g_0(\nu)-g_0(-\nu)=0. \tag{4.15}$$

In our example, this is in fact the case; Eq. (4.12) has no $O(1/k)$ term, and g_0 fits Eq. (4.15).

Restrictions on the $f_0(\nu)$ follow from the consideration of the Wronskian W. We have

$$W(w(\nu,r),w(-\nu,r))=-2\nu=W(u(\nu,r),u(-\nu,r)). \tag{4.16}$$

To $O(g^2)$, Eq. (4.13) shows that this requires

$$F(\nu)+F(-\nu)=0$$

or

$$f_i(\nu)+f_i(-\nu)=0. \tag{4.17}$$

This is a nontrivial restriction; analogous restrictions on the $g_i(\nu)$ do not exist to $O(g^2)$, and consideration of the $O(g^4)$ results would be required to find them. Our example is consistent with Eq. (4.17). The restrictions provided by the Wronskian show that $\hat{S}$ of the form of Eq. (2.4), obeying (2.5), is for our purposes unique. To see this, suppose $u(\nu,r)\to\bar{u}(\nu,r)=h(\nu^2)u(\nu,r)$, a shift which leaves $\hat{S}=u(-\nu,R)/u(\nu,R)$ invariant. By our Wronskian condition, however,

$$W(u(\nu,r),u(-\nu,r))$$
$$=W(\bar{u}(\nu,r),\bar{u}(-\nu,r))$$
$$=h^2(\nu^2)W(u(\nu,r),u(-\nu,r))$$

or

$$h^2(\nu^2)=1.$$

To see how all this works, let us consider a simple example, namely, suppose the asymptotic behavior of the two-point function for $\nu=\frac{1}{2}$ is

$$\hat{S}_A=(-ik)\left[1+\frac{a}{k^2}+\frac{ib}{k^3}+\cdots\right]. \tag{4.18}$$

We consider this as the input, and we shall assume the confining potential gives linear (or more slowly rising) asymptotic trajectories. This last assumption means that the confining potential behaves as r^{2n}, $n \geq 1$, or equivalently that $\hat{S}_0$ behaves at large k as

$$\hat{S}_0=(-ik)\left[1+O\left[\frac{1}{k^{2n+2}}\right]\right], \quad n\geq 1 \ .$$

Thus a and b determine the constant and linear terms in the small-r expansion of the perturbing potential. Thus a and b determine the strength g^2 and range μ of a perturbing potential like $g^2=e^{-\mu r}$. In fact, from Eq. (4.12), we see

$$g^2=2a, \quad \mu=-\frac{2b}{a} \ .$$

To determine the shift in pole positions, we need to know $G(\nu)$ at finite values of k^2. We take these from the (constant) values given in Eq. (4.10). Physically this is reasonable because the shift in a bound state due to a short-range perturbing potential cannot depend on anything other than the behavior of the confining potential near the origin. In our example, then, from Eq. (4.10)

$$G(\tfrac{1}{2})=2 \ .$$

Equation (4.13) now gives us the shifted pole positions and residues. For example, we could find the shifted poles to $O(g^2)$ from Eq. (3.14), or better, we could work directly with $\hat{S}$ as in Eq. (4.11), whose poles at $E_n^{(1)}$ are determined by the zeros of

$$D=1-\frac{g^2}{\mu^2+4k^2}+\hat{S}_0\frac{2}{\mu}\frac{g^2}{\mu^2+4k^2} \ . \tag{4.19}$$

To find the new energy levels $E_n^{(1)}$ we expand $\hat{S}_0$ around the unperturbed levels $E_n^0=k_n^{02}$. We have

$$\hat{S}_0^{-1}(E_n^{(1)})=\hat{S}^{-1}(E_n^0)+(E_n^{(1)}-E_n^0)\frac{d}{dZ}\hat{S}_0^{-1}(Z)\Big|_{Z=E_n^0}$$

or (4.20)

$$\hat{S}_0(E_n^{(1)})=\frac{1}{E_n^{(1)}-E_n^0}\frac{1}{(d/dZ\hat{S}_0{}^{-1}(Z)\,|_{Z=E_n^0}} \ .$$

Substitute Eq. (4.20) into (4.19) with $D=0$, substituting $E_n=E_n^{(0)}$ everywhere except (4.20), and expand to $O(g^2)$. The result is

$$E_n^{(1)}=E_n^0+\frac{2}{\mu}\frac{g^2}{\mu^2+4E_n^0}\frac{1}{(d/dZ\hat{S}_0{}^{-1}(Z)\,|_{Z=E_n^0}} \ . \tag{4.21}$$

Equation (4.21) gives us for our example of the exponential in a long-range oscillator

$$E_n^{(1)}=E_n^0-\frac{2a}{4b^2/a^2+4E_n^0}\frac{4\omega}{\pi\mu}\frac{\Gamma(\frac{3}{4}+E_n^0/2\omega)}{\Gamma(\frac{1}{4}+E_n^0/2\omega)}\sqrt{2\omega} \ , \tag{4.22}$$

where we recall [Eq. (2.9)] that

$$E_n^0=2\omega(\tfrac{3}{4}+n) \ .$$

This can be compared with the attractive short-range exponential in the cavity, for which

$$E_n^{(1)}=E_n^0-\frac{2a}{4b^2/a^2+4E_n^0}\frac{4}{\mu R}E_n^0 \ , \tag{4.23}$$

where

$$E_n^0=\frac{\pi^2 n^2}{R^2} \ .$$

For both cases (long-range cavity and long-range harmonic oscillator) the behavior of $\hat{S}_A$ is identical to $O(1/k^3)$; the differences in the pole positions—Eqs. (4.22) and (4.23)—come from one's understanding of the confining force, which leads to different $\hat{S}_0$. For the cavity $\hat{S}_0$ was $k\cot R$, and for the oscillator $\hat{S}_0$ was

$$\frac{\Gamma(\frac{3}{4}-E/2\omega)}{\Gamma(\frac{1}{4}-E/2\omega)}\sqrt{2\omega} \ .$$

Our ansatz represents a kind of twist on effective-energy theory. We use the *high-energy* behavior to determine two parameters with dimensions of length, namely, μ^{-1} and g^{-1}. These parameters in turn affect pole positions. In phrasing our ansatz as an effective-range theory we want to emphasize that use of a Schrödinger equation with a definite potential is not an essential feature.

V. CONCLUSIONS

We have investigated the properties of a nonrelativistic two-point function which contains information both on the bound states of a confined system and on asymptotic behavior of that system. The Green's function used in studying earlier sum rules[5] is a special S-wave case of this two-point function.

This two-point function is completely equivalent to an analog to the S matrix, $\hat{S}$, for confined systems. Since we have shown that $\hat{S}$ is a summation over the poles of the system, and since $\hat{S}$ transparently displays the asymptotic behavior, we are led to a constructive program for understanding the bound states of QCD. Both experiment and many studies of confinement in QCD suggest that bound quarks lie on linear Regge trajectories. This gives us, using the results of this and earlier papers, a zeroth-order starting point both for the bound-state poles and the asymptotic behavior specified by $\hat{S}_0$. But we can independently calculate the asymptotic two-point functions in QCD using perturbation theory, thanks to asymptotic freedom. To the extent this behavior differs from that of $\hat{S}_0$, we modify the properties of the low-lying bound states. Such modification is a manifestation of short-range forces between the constituents.

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